ON THE DYNAMICS IN A COBWEB MOEL WITH ADAPTIVE PRODUCTION ADJUSTMENT

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Abstract:
We firstly make a short description of de concept of difference equation, equilibrium points, classification and the cobweb diagram. We will present an economical example for better understanding these concepts. Then we will simplify the cobweb model present by [OnSiYo]. We will analyse the new model and we will present some graph representations by using Mapple for sustain the theoretical results. We also suggest another model which can be analyzed in the same way.

Key words: *Firs-order difference equation, cobweb diagram, fixed point.*

JEL classification: *C44*

1. Introduction

Difference equations describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the (n+1)th generation $x(n+1)$ is a function of the nth generation $x(n)$. This relation expresses itself in the *difference equation*

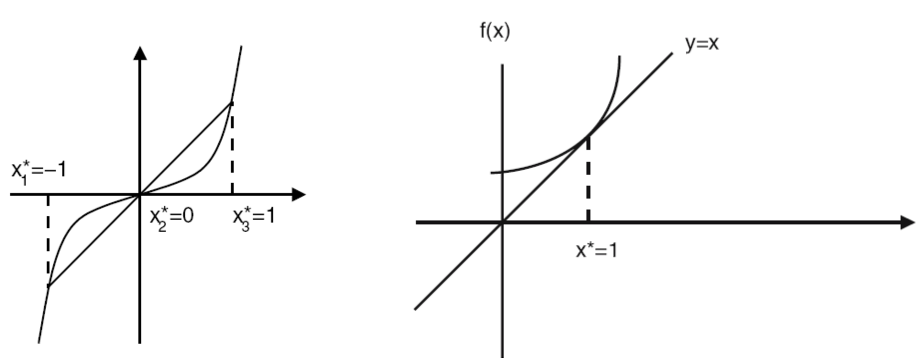
$$x(n+1) = f(x(n)) \tag{1}$$

We consider then $O(x_0) = \{f^n(x_0) : n \geq 0\} = \{x_0, f(x_0), \cdots, f^n(x_0)\}$ the (*positive*) *orbit* of x_0, where $f^n(x_0)$ is the nth iterate of x_0 under f. This iterative procedure is an example of a *discrete dynamical system*. Letting $x(n) = f^n(x_0)$, we have $x(n+1) = f^{n+1}(x_0) = f[f^n(x_0)] = f(x(n))$.

In many applications in economics it is desirable that all solutions of a given system tend to its equilibrium point. We now give the formal definition of an equilibrium point.

Definition 1.1 (Elaydi, 2005) A point x^* in the domain of f is said to be an *equilibrium point* of equation (1) if it is a fixed point of f, i.e., $f(x^*) = x^*$.

Graphically, an equilibrium point is the x-coordinate of the point where the graph of f intersects the diagonal line $y = x$ (Figure 1.1).



(a) (b)

Figure 1: (a) Fixed points of $f(x) = x^3$; (b) Fixed points of $f(x) = x^2 - x + 1$.

It is possible in difference equations that a solution may not be an equilibrium point but may reach one after finitely many iterations. In other words, a nonequilibrium point may go to an equilibrium point in a finite time. This leads to the following definition.

Definition 1.2.(Elaydi, 2005) Let x be a point in the domain of f. If there exists a positive integer r and an equilibrium point x^* of equation (1) such that $f^r(x) = x^*$, $f^{r-1}(x) \neq x^*$, then x is an *eventually equilibrium (fixed) point*.

One of the main objectives in the study of a dynamical system is to analyze the behavior of its solutions near an equilibrium point. This study constitutes the stability theory. Next we recall the basic definitions of stability.

Definition 1.3.(KL)

(a) The equilibrium point x^* of equation (1) is *stable* (Figure 2) if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies $|f^n(x_0) - x^*| < \varepsilon$ for all $n > 0$. If x^* is not stable, then it is called *unstable* (Figure 2.a).

(b) The point x^* is said to be *attracting* if there exists $\eta > 0$ such that $|x(0) - x^*| < \eta$ implies $\lim_{n\to\infty} x(n) = x^*$. If $\eta = \infty$, x^* is called a *global attractor* or *globally attracting* (Figure 2.b).

(c) The point x^* is an *asymptotically stable equilibrium point* if it is stable and attracting. If $\eta = \infty$, x^* is said to be *globally asymptotically stable* (Figure 2.c).

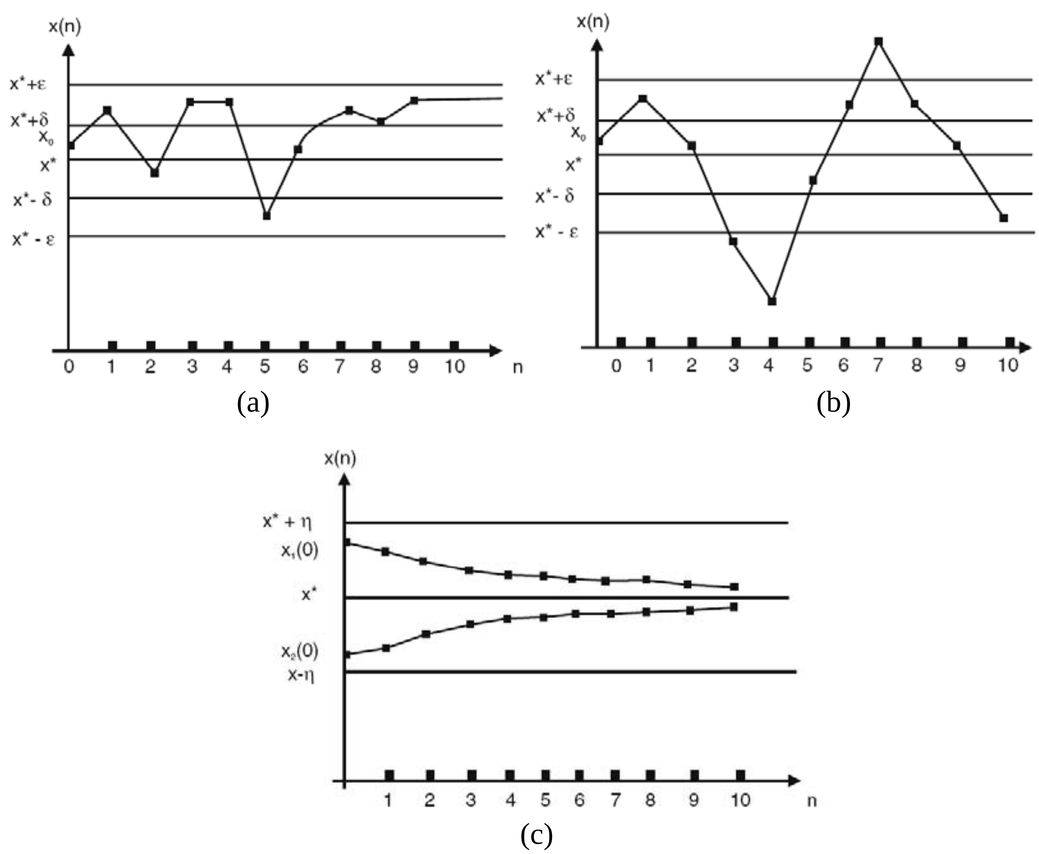


Figure 2: (a) Stable x^*. (b) Unstable x^*. (c) Asymptotically stable x^*.

To determine the stability of an equilibrium point from the above definitions may prove to be a mission impossible in many cases. This is due to the fact that we may not be able to find the solution in a closed form even for the deceptively simple-looking equation (1). In section 2 we highlight a graphical technique that allows us to understand the behavior of solutions of equation (1) in the neighbourhood of equilibrium points.

2. The Stair Step (Cobweb) Diagrams

An important graphical method for analyzing the stability of equilibrium (and periodic) points for equation (1) is *the stair step (cobweb) diagrams*.

Since $x(n+1) = f(x(n))$, we may draw a graph of *f* in the $(x(n), x(n+1))$ plane. Then, given $x(0) = x_0$, we pinpoint the value x_1 by drawing a vertical line through x_0 so that it also intersects the graph of *f* at (x_0, x_1). Next, draw a horizontal line from (x_0, x_1) to meet the diagonal line $y = x$ at the point (x_1, x_1). A vertical line drawn from the point (x_1, x_1) will meet the graph of *f* at the point (x_1, x_2). Continuing this process, one may find $x(n)$ for all $n > 0$.

Example 2.1. (Eladyi, 2005) *The Cobweb Phenomenon* – the pricing of a certain commodity.

Let:

- $S(n)$ - the number of units supplied in period *n*;
- $D(n)$ - the number of units demanded in period *n*;
- $p(n)$ - the price per unit in period *n*;
- $m_d > 0$ - the sensitivity of consumers to price;
- $m_s > 0$ - the sensitivity of suppliers to price.

We assume that $D(n)$ depends only linearly on $p(n)$ and is denoted by $D(n) = -m_d p(n) + b_d, b_d > 0$ (the price-demand curve) (2)

We assume that the price–supply curve relates the supply in any period to the price one period before, i.e. $S(n+1) = m_s p(n) + b_s, b_s > 0$ (3)

The slope of the demand curve is negative because an increase of one unit in price produces a decrease of m_d units in demand. Correspondingly, an increase of one unit in price causes an increase of m_s units in supply, creating a positive slope for that curve.

We assume that the price is the price at which the quantity demanded and the quantity supplied are equal, that is, at which $D(n+1) = S(n+1)$.

Thus $-m_d p(n+1) + b_d = m_s p(n) + b_s \Rightarrow p(n+1) = Ap(n) + B = f(p(n))$, (4)

$$\text{where } A = -\frac{m_s}{m_d}, B = \frac{b_d - b_s}{m_d}. \quad (5)$$

This equation is a first-order linear difference equation. The equilibrium price p^* is defined in economics as the price that results in an intersection of the supply $S(n+1)$ and demand $D(n)$ curves. Also, since p^* is the unique fixed point of $f(p)$ in (4), $p^* = \frac{B}{1-A}$. Because A is the ratio of the slopes of the supply and demand curves, this ratio determines the behavior of the price sequence. There are three cases to be considered (the three cases are simultaneous depicted graphically using the stair step diagram):

(a) $-1 < A < 0$:

- In this case, prices alternate above and below but converge to the equilibrium price p^*. In economics lingo, the price p^* is considered “stable”; in mathematics, we refer to it as “asymptotically stable” (Figure 3.a).

(b) $A = -1$:

- In this case, prices oscillate between two values only. If $p(0) = p_0$, then $p(1) = p_0 + B_0$ and $p(2) = p_0$. Hence the equilibrium point p^* is stable (Figure 3.b).

(c) $A < -1$:

- In this case, prices oscillate infinitely about the equilibrium point p^* but progressively move further away from it. Thus, the equilibrium point is considered unstable (Figure 3.c).

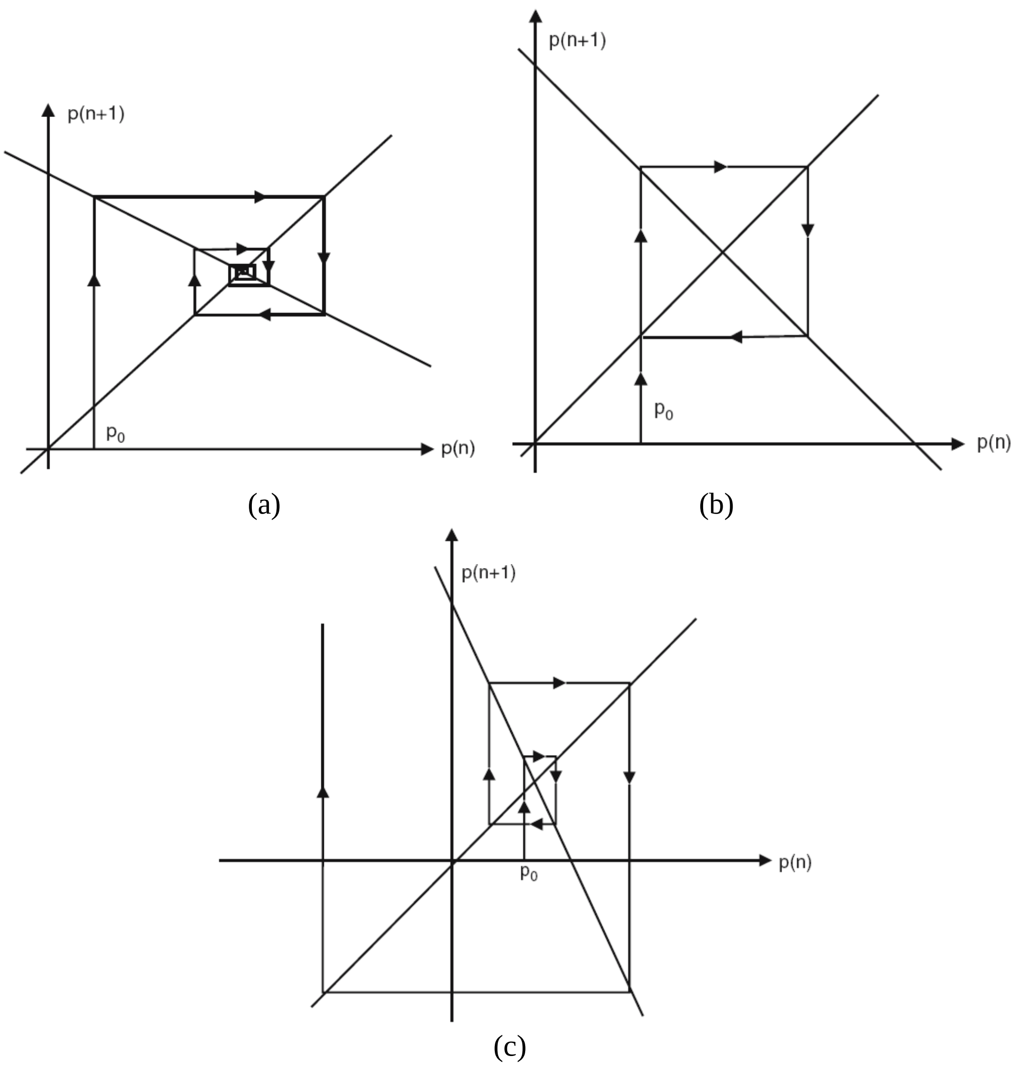


Figure 3: (a) Asymptotically stable equilibrium price; (b) Stable equilibrium price; (c) Unstable equilibrium price.

An explicit solution of (4) with $p(0) = p_0$ is given by $p(n) = \left(p_0 - \frac{B}{1-A}\right)A^n + \frac{B}{1-A}$.

2. Application

2.1. Its model

Our starting point is a recent economical problem and his model described in [OnSiYo]. The aim of our approach is a better understanding of the behaviour of a new type of cobweb model.

The problem: A farmer has to decide how much to produce in a certain period before price is determined and sales revenues are received.

The model: - at period *t*, a supplier decides his production x_{t+1} for period *t*+1. As he knows well, even a production plan that maximizes profits may turn out to be a disaster in reality. He calculates the profit maximum $\tilde{x}_{t+1}$ and uses it as a target of adjustment. The calculation is done relative to the quadratic cost function $\frac{b}{2}x^2, b > 0$ and naive price expectation, which means that his price expectation for the next period is equal to the current price p_t.

Thereby, in the [OnSiYo] page 103, the model of this problem can be reduced to the following first-order difference equation

$$z_{t+1} = (1-\alpha)z_t + \frac{\alpha}{z_t^{\beta}},$$

where $\alpha \in (0,1)$ is the speed of adjustment and $\frac{1}{\beta}, \beta > 0$ the constant price elasticity.

Next, we will present two slightly different adjustment models for this problem: the first model is a simplified one and the second is a generalized model.

2. 2 The new model.

We will consider the following first-order difference equation:

$$z_{t+1} = (1-\alpha)z_t + \frac{\beta}{1+z_t} \tag{6}$$

In the initial model, we have replaced $\frac{\alpha}{z_t^{\beta}}$ with $\frac{\beta}{1+z_t}$. Notice that both functions are decreasing to 0 at infinity.

Analysis of the model

Our model (6) can be reformulated by the two-parameter family of functions $f_{\alpha,\beta} : R_+^* \to R_+^*$ as

$$f_{\alpha,\beta}(x) = (1-\alpha)x + \frac{\beta}{1+x}, \alpha \in (0,1), \beta \in (0,\infty).$$

which, for simplicity, will be expressed as *f*.

First we will find the fixed point for *f* using for the computation the Maple 9 software, i.e.: $f(x^*) = x^* \Leftrightarrow x^* = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\frac{\beta}{\alpha}}$. There are two fixed points of *f*, but only one is positive.

We calculated the first and second derivatives by using the Maple software and we obtained:

$$f'(x) = 1 - \alpha - \frac{\beta}{(1+x)^2}, \quad f''(x) = \frac{2\beta}{(1+x)^3} > 0, x \in R_+^*,$$

which show us that *f* is a strictly convex and unimodal function on R_+^*.

We calculate then the value of the first derivative in the fixed point:

$$f'(x^*) = 1 - \alpha - \frac{4\beta}{\left(1 + \sqrt{1 + \frac{4\beta}{\alpha}}\right)^2} = \varphi(\alpha, \beta)$$

The graph of function *f* for some values of α and β is presented in Figure 4.

If $f'(x^*) \in (-1,1)$ then there exists a closed interval V_{x^*} of x^* such that $|f'(x)| < l < 1$ on V_{x^*}, and we obtain that $|f(x) - x^*| \le l|x - x^*|$; this implies that x^* is an attractive point. More exactly,

1. V_{x^*} is invariant, i.e. $f(V_{x^*}) \subset V_{x^*}$;
2. *f* is a strict contraction[1] on V_{x^*}.

Using the Banach[2] fixed point theorem we obtain that x^* is a fixed point locally attractive.

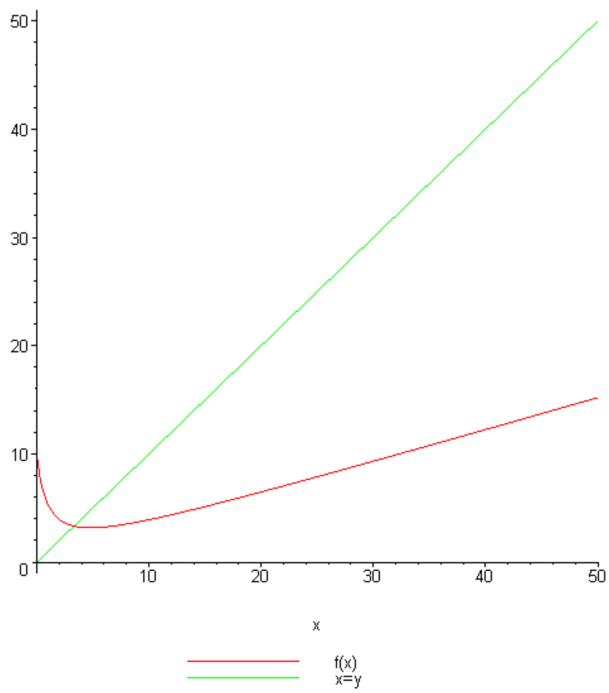


Figure 4: Graph of *f* for $\alpha = 0.7$ and $\beta = 10$

We evaluate $f'(x^*)$ for α from 0.01 to 0.99 and for β from 0 to 400 (see Figure 5).We observes in Figure 5 that for different values of α and β the

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[1] **Definition:** $f : X \to X$, *f* is a strict *contraction* if there exist a $\alpha \in (0,1)$ such that $d(f(x), f(y)) < \alpha d(x, y), \forall x, y \in X$.

[2] Theorem (Classical Banach fixed point theorem): Let (X, d) be a complet metric space. If $f : X \to X$ is a strict contraction then there exist a unique $x^* \in X$ such that $f(x^*) = x^*$. Moreover x^* star is the limit of successive approximations, starting at any point of *X*.

value $|f'(x^*)| < l$ is always under 1 this implies that x^* is a globally attractive fixed point and $V^* = (0, \infty)$ is its domain of attraction.

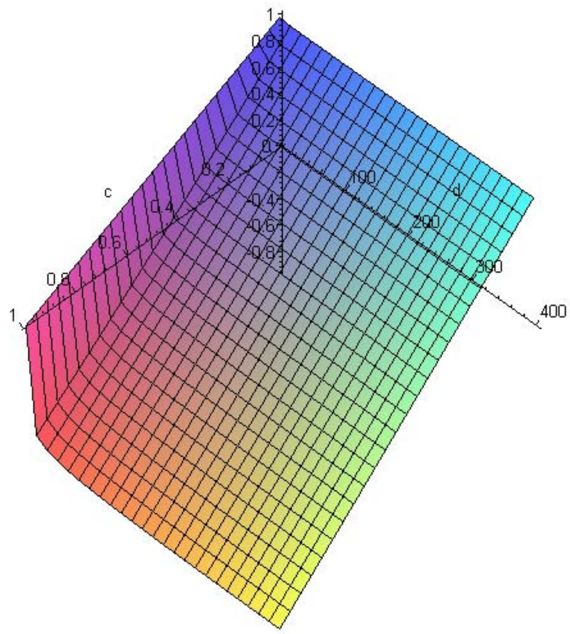
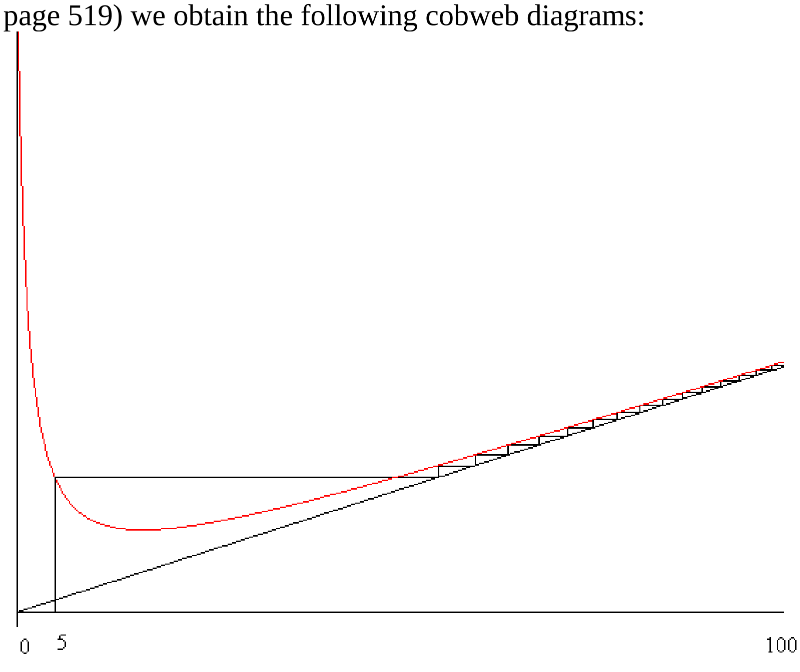
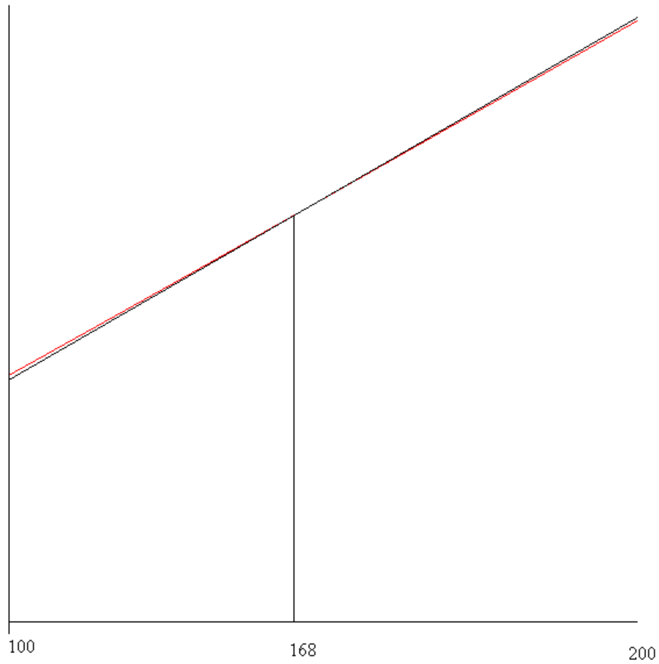


Figure 5: The values of $f'(x^*)$, where $\begin{matrix} c = \alpha \in (0,1) \\ d = \beta \in (0,400) \end{matrix}$

Using the "`Cobweb Program`" which is implemented in Maple 9 (see [Elaydi], page 519) we obtain the following cobweb diagrams:



(a)



(b)

Figure 5: Cobweb diagrams for f

(a) $x_0 = 0.1, \alpha = 0.01, \beta = 300$

(b) $x_0 = 168, \alpha = 0.01, \beta = 300$

If we set α very little and set β to a big value we have the impression that the sequence tend to infinity (see Figure 5.a). In reality, the fixed point is very big (x^*=172.7) and if we set $x \in (100,200)$ and $x_0 = 168$ we observe that the graph of f (the red line in Figure 5.b) go from upper de first bisector under the first bisector. This mine that f have a fixed point.

A discussion on the parameters α and β

We discus now the behavior of parameters α, β on the curve $d = -\dfrac{c^2(-1+c)}{(-1+2c)^2}$ (we have obtaining this curve by solving $f'(x^*) = 0$).

Case 1: $x_{\min} < x^*$

Let $\gamma < x_{\min}$ with $f(\gamma) = x^*$

1. If $x_0 \in (x^*, \infty)$ then the sequence x_n is decreasing to the fixed point.
2. If $x_0 < \gamma$ then $x_1 > x^*$ and the sequence x_n is decreasing to the fixed point.
3. If $\gamma < x_0 < x_{\min}$ hen $x_1 \in (x_{\min}, x^*)$ and the sequence x_n is increasing to the fixed point.

Case 2: $x^* < x_{\min}$

We observe that the sequence x_n oscillates and converges to the fixed point (see Figure).

Case 3: $(\alpha, \beta) \in$ the curve $d = -\dfrac{c^2(-1+c)}{(-1+2c)^2}$; then $x_{\min} = x^*$ (see Figure 6)

From x_1 the graph decreases to the fixed point x^*, regardless of the x_0 position.

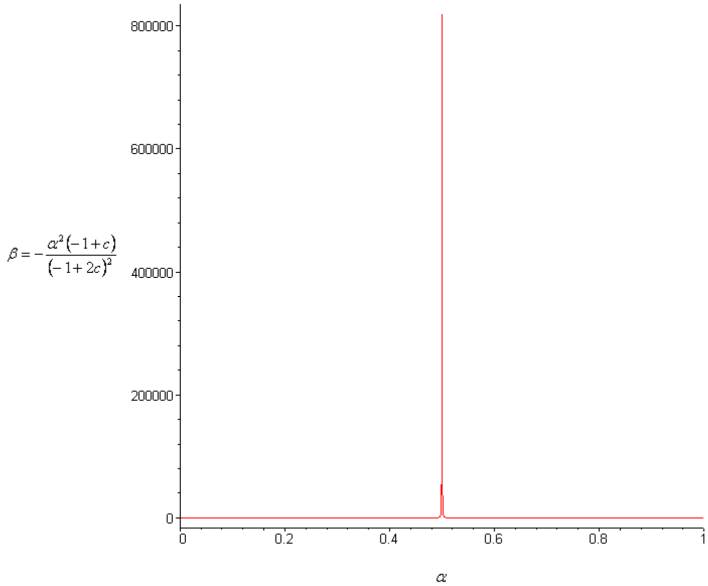
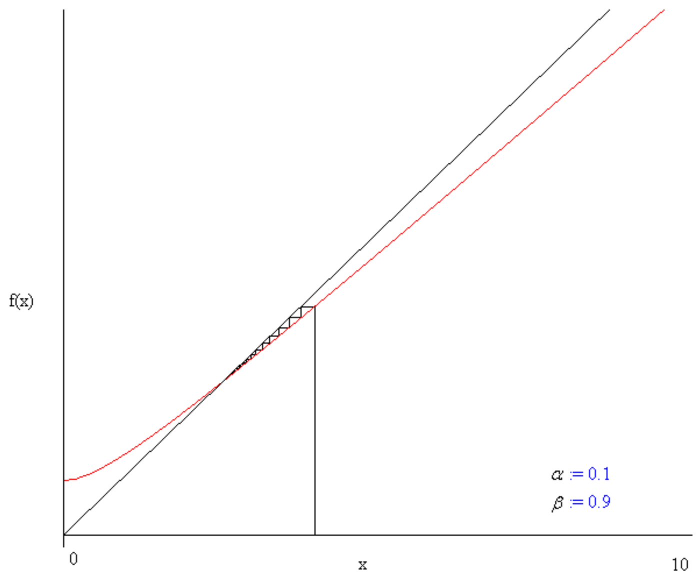
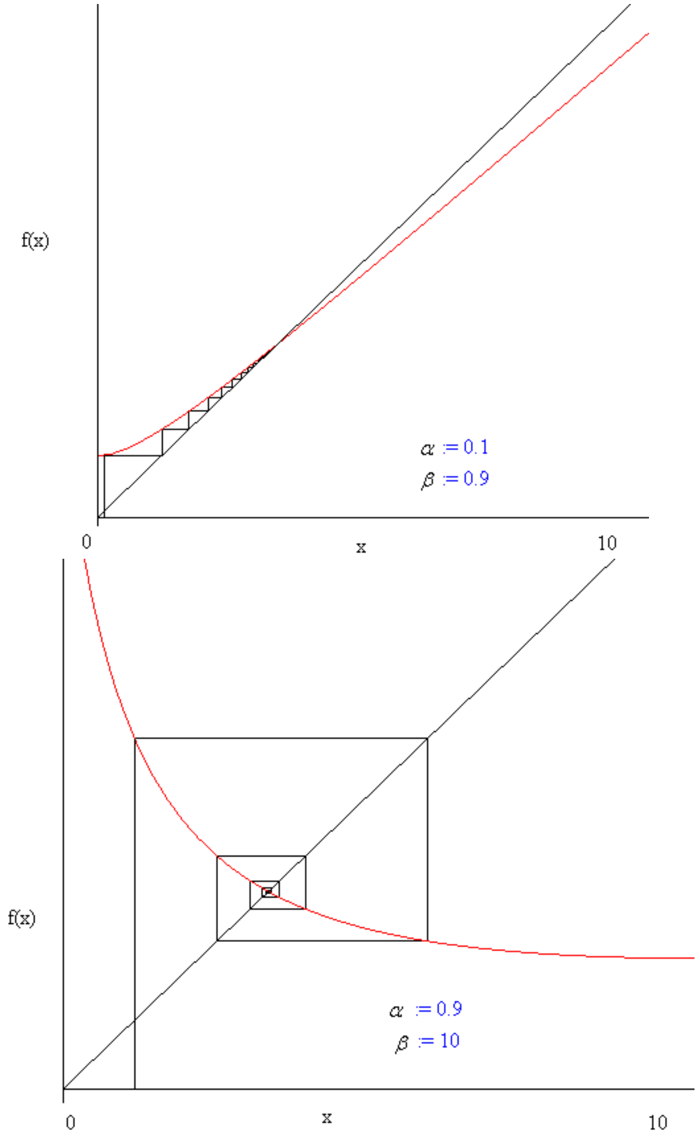


Figure 6: Graph of the curve β

In the following figures, for different values of parameters α and β, the cobweb diagrams are presented and, as a conclusion, we see that x^* is an asymptotically stable equilibrium.





Figures 7, 8, 9: The cobweb diagrams for *f*

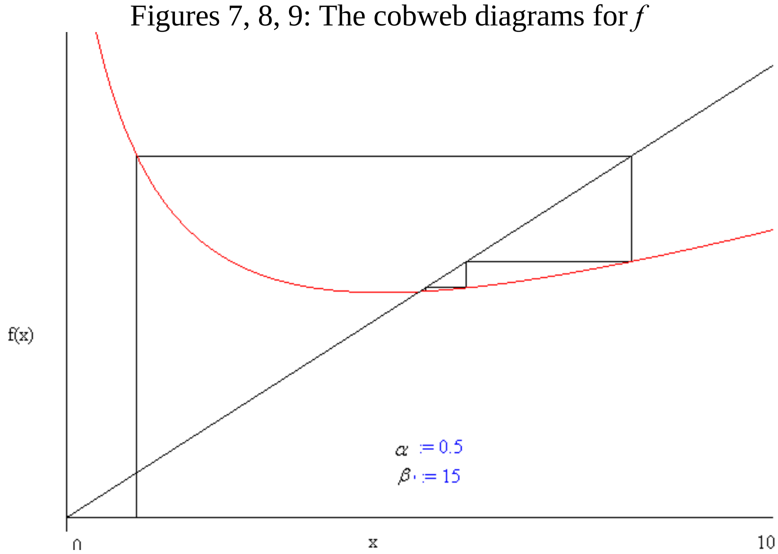


Figure 10: The cobweb diagram for *f*

Remark. Except for obvious modifications, the above method can be used to study the following generalized model of the initial problem:

$$z_{t+1} = (1-\alpha)z_t^{\beta} + \frac{\beta}{z_t^{\beta}}.$$

4. Conclusions

For analysing the cobweb model, the following notions have been used:

- the monotony: the behaviour of the recurrent sequences is precisely determined for monotone functions;
- the contraction mappings and the classical Banach fixed point theorem;
- the cobweb diagrams and there implementation by using the Maple 9 software (easyer computations and have graphical representation of the cobweb diagrams).

Acknowledgment. We have indebted to Prof. V. Radu for useful discussions.

REFERENCES

[Elaydi] S. Elaydi, *An Introduction to Difference Equations. Third Edition,* Springer, 2005;

[KL] Klaus Neusser, *Difference Equations for Economists, 2008;*

[OnSiYo] T. Onozaki, G. Sieg, M. Yokoo, *Complex Dynamics in a Cobweb Model with Adaptive Production Adjustment,* Journal of Economic Behavior& Optimization, Vol. 41(2000) 101-115;

[OnSiYo2] T. Onozaki, G. Sieg, M. Yokoo, *Stability, chaos and multiple attractors: a single agent makes a difference*, Journal of Economic Dynamics& Control, Vol. 27(2003) 1917-1938;

[VRadu] V. Radu, *LECȚII de teoria punctelor fixe*, UVT, 1998.