STATIONARY SEQUENCES OF DISTRIBUTION ON HILBERT SPACE

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Abstract:
In this paper we develop the correlation theory of a stationary process of distribution and culminating in the now classical Wold decomposition theorem. The fundamental notion of correlation theory of a stationary process of distribution such as distribution, sequence of distribution, normal and orthogonal process, process of innovation, deterministic are define in this context.

***Key words**: stationary process, distribution, Wold decomposition*

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Preliminaries

K will stand for a separable complex Hilbert space. Let (Ω, B, P) be a probability space. We write K' for the complex Hilbert space $L^2(\Omega, B, P)$. The inner products in K, K' will be denoted booth by $(.,.)$ the risk of confusion being hereby small.

Definition 1. *A **distribution** on K is a continuous linear map from K to K'.*

In this paper familiar mathematical object is given a new name. One reason is that for some purposes it is useful to consider certain equivalence classes of distribution [4], and in doing this the structure of the probability space underlying K' is used more crucially than we have indicated.

Definition 2. *Let F, G be the distribution on K. For $y, z \in K$ consider the continuous conjugate bilinear functional (Fy, Gz). By the Riesz theorem there is a **bounded operator** $\langle G, F\rangle : K \to K$ such that $(Fy, Gz) = (\langle G, F\rangle y, z)$. We call $\langle G, F\rangle$ the Gramian of F, G.*
It is clear that $\langle G, F\rangle = \langle F, G\rangle^*$ and that $\langle F, F\rangle$ is a non-negative operator.

Definition 3. *Let $\{F_n\}_{-\infty}^{+\infty}$ be a sequence of distribution. We say*

(a) $F_i \perp F_i$ *if* $\langle F_i, F_i\rangle = 0$;

(b) F_i *is **normal** if* $\langle F_i, F_i\rangle = I$;

(c) *The sequence* $\{F_n\}_{-\infty}^{+\infty}$ *is **orthogonal** if* $i \neq j \Rightarrow F_i \perp F_j$;

(d) *The sequence is **orthonormal** if* $\langle F_i, F_j\rangle = \delta_{ij} I$.

Definition 4. *By D we shall denote the set of all distributions, endowed with the normal topology of the space of continuous linear maps $K \to K'$.*

Definition 5. *(i) A linear manifold in D is a non-void subset $M \subset D$ such that if $F_1, F_2 \in M$, $A_1, A_2 \in B(K)$ than $F_1A_1 + F_2A_2 \in M$.*

(ii) A subspace of D is a closed linear manifold.

Definition 6. *$\sigma(F_n)_{-\infty}^{+\infty}$ will stand for the minimal subspace containing $\{F_n\}_{-\infty}^{+\infty}$.*

Definition 7. *If $(M_j)_{j\in J}$ is a family of subspaces of D, then by $\sum_{j\in J} M_j$ we mean the set of all sums $\sum_{j\in J} F_j$ which conserve in D with $F_j \in M_j$.*

Using the above definitions, we can now check the following properties.

(1) Let $F_j, G_k \in D$, $A_j, B_k \in B(K)$, $j,k = 1,...,n$. Then

$$\left\langle \sum_{k=1}^{n} G_k B_k, \sum_{j=1}^{n} F_j A_j \right\rangle = \sum_{k=1}^{n}\sum_{j=1}^{n} B_k^* \langle G_k, F_j \rangle A_k .$$

(2) If $F, G \in D$ then $F \perp G$ if and only if $(Fe_i, Ge_i) = 0$ where $(e_i)_1^{+\infty}$ is a complete orthonormal set in K.

(3) $\{F_j\}_{j\in J}$ is an orthonormal set in D if and only if the set $\{F_j e_k \mid j \in J, k = 1,2,...\}$ is an orthonormal set in K', $(e_i)_1^{+\infty}$ being as in (2).

(4) Let $\{F_n\}_{-\infty}^{+\infty}$ be a sequence in D such that $\langle F_m, F_n \rangle = \delta_{mn} T_n$ where T_n is a non-negative operator in $B(K)$ for each n. If T_n is invertible in $B(K)$ for each n, then $F_n T_n^{-1/2}$ is an orthonormal sequence in D.

(5) $M \subset D$ is a closed subspace of D if and only if there is a closed subspace M' of K' such that $M = \{F \mid F \in D, \textit{and } Fx \in M' \textit{ for each } x \in K\}$. We call this subspace M' the subspace associated with M.

(6) Let M be a closed subspace of D and $F \in D$. Then there is a unique element $(F \mid M)$ of M such that $\langle G, F - (F \mid M) \rangle = 0$ for each $G \in M$. $(F \mid M)$ shall be called the orthogonal projection of F on M. In fact, if $x \in K$, $(F \mid M)x = P_{M'} Fx$ where $P_{M'}$ is the orthogonal projection on M' in K'.

(7) Let M, N be the subspaces of D with $M \subset N$, then clearly $M' \subset N'$. Then there is a unique subspace $N \Theta M$ of D such that $M \oplus (N \Theta M) = N$ and $M \perp (N \Theta M)$. In fact $N \Theta M$ is the unique subspace of D whose associated subspace is $N' \Theta M'$, the orthogonal complement of M' in N'.

(8) Let M be a subspace of D and $F \in D$. If $G \in M$ then we have

$$\langle F - (F \mid M), F - (F \mid M) \rangle \prec \langle F - G, F - G \rangle$$

where $A \prec B$ means that $B - A$ is non-negative.

(9) If M, N are the subspaces of D such that $M \perp N$ then $M \oplus N$ is a subspace of D and if $F \in D$ then

$$(F \mid M + N) = (F \mid M) + (F \mid N) .$$

(10) If M, N are the subspaces of D such that $M \subset N$ and if $F \in D$ then

$$\langle (F \mid M), (F \mid M) \rangle \prec \langle (F \mid N), (F \mid N) \rangle .$$

(11) If $M_k k = 0, \pm 1, \pm 2, ...$ are subspaces of D such that $M_k \supset M_{k+1}$ and if $M = \bigcap_{-\infty}^{+\infty} M_k$ then if $F \in D$, we have

$$(F \mid M) = \lim_{k \to +\infty} (F \mid M_k) .$$

The Wold decomposition

Let $\{F_n\}_{-\infty}^{+\infty}$ be a stationary sequence of distributions on K. We set M'_∞ the closure in K' of the union of the ranges of $\{F_n; n = 0, \pm 1, \pm 2, \ldots\}$ and call this the universe of our process. It is clearly a separable subspace of K' and is easily seen to be the associated subspace of the subspace $\sigma(F_n)_{-\infty}^{+\infty}$ of D.

We define an operator U on elements of M'_∞ of the form

$$\sum_{j=-N}^{N}\sum_{k=-M}^{M} a_{jk} F_j e_k \,,$$

where N, M are integers and the e_k are arbitrary elements of K by

(12) $$U\left(\sum_{j=-N}^{N}\sum_{k=-M}^{M} a_{jk} F_j e_k\right) = \sum_{j=-N}^{N}\sum_{k=-M}^{M} a_{jk} F_{j+1} e_k \,.$$

U clearly maps the linear manifold of elements of M'_∞ of the above form onto itself in a one-one manner. Further , the assumption that $\{F_n\}_{-\infty}^{+\infty}$ is stationary shows, after an easy computation, that U preserves inner products. Since elements of the form above are dense in M'_∞, U can be extended to a unitary operator on K' which maps M'_∞ onto itself. While that is true that the extension is not unique, any two extension have clearly the same restriction to M'_∞. From now on we can and do assume that $K' = M'_\infty$ so that U is uniquely defined on K'. U is called the shift operator of the process.

Let us write

$$M_k = \sigma(F_n)_{-\infty}^{k}\,,\; M_{-\infty} = \bigcap_{-\infty}^{+\infty} M_k$$

and call these subspaces of D “the universe of the process up to time k” and respectively “the remote past of the process”. It is easy to see that

$$U^n M_k = M_{k+n}$$

$$U^n (F_j \mid M_k) = (F_{j+n} \mid M_{k+n})$$

Definition 8. *The process of distribution* $\{G_n\}_{-\infty}^{+\infty}$ *defined by*

$$G_n = F_n - (F_n \mid M_{n-1})$$

is called the ***process of innovation*** *of* F_n. *It is clear that* $\{G_n\}_{-\infty}^{+\infty}$ *is stationary,* $G_n = U^n G_0$ *and further that* $\langle G_{m,} G_n \rangle = \delta_{mn} S$, *where* S *is a non-negative operator independent of* n.

Definition 9. $\{F_n\}_{-\infty}^{+\infty}$ *is said to be* ***purely deterministic*** *if* $G_n = 0$ *for all* n. *If not we call non-deterministic.*

The operator $S = \langle G_n, G_n \rangle$ will be called the **prediction error operator** of $\{F_n\}_{-\infty}^{+\infty}$. It is a bounded non-negative operator and is zero if and only if $\{F_n\}_{-\infty}^{+\infty}$ is purely deterministic.

Definition 10. *We say that* $\{F_n\}_{-\infty}^{+\infty}$ *has* ***nearly full rank*** *if* S *has trivial null space and it is said to have* ***full rank*** *if* S *admits bounded inverse.*

The following theorem is preliminary to the Wold decomposition.

Theorem 1. *Let* $\{G_n\}_{-\infty}^{+\infty} \in D$ *such that* $\langle G_{m,} G_n \rangle = \delta_{mn} S$ *where* $S \neq 0 \in B(K)$. *Then*

(a) $\sigma(G_n) \perp \sigma(G_m)$ *if* $n \neq m$;

(b) *If* $H = \sum_{n=-\infty}^{+\infty} H_n$, $K = \sum_{n=-\infty}^{+\infty} K_n \in D$ *with* $H_n, K_n \in \sigma(G_n)$, *then*

$$\langle H, K \rangle = \sum_{n=-\infty}^{+\infty} \langle H_n, K_n \rangle ;$$

(c) $\sum_{-\infty}^{+\infty} \sigma(G_n) = \sigma(G_n)_{-\infty}^{+\infty}$;

(d) *If* S *is invertible, then every element of* $\sigma(G_n)$ *is of the form* $G_n A$ *with* $A \in B(K)$ *;*

(e) *If* $K \in D$, *then there exist* $K_n \in \sigma(G_n)$ *such that*

$$\left(K \mid \sigma(G_n)_{-\infty}^{+\infty}\right) = \sum_{n=-\infty}^{+\infty} K_n \text{ and } \langle G_n, K \rangle = \langle G_n, K_n \rangle .$$

Further, if S *is invertible then there exist* $A_n \in B(K)$ *such that* $K_n = G_n A_n$ *and* $\langle G_n, K \rangle = SA_n$.

Theorem 2. ***(Wold decomposition****) Let* $\{F_n\}_{-\infty}^{+\infty}$ *be a stationary process in* D *and let* $\{G_n\}_{-\infty}^{+\infty}$ *be its innovation process. Let* M_n, N_n *be the universes up to time* n *of* $\{F_n\}_{-\infty}^{+\infty}$ *and respectively* $\{G_n\}_{-\infty}^{+\infty}$. *Then we have*

(i) $F_n = P_n + Q_n$ *where* $P_n = (F_n \mid N_n)$, $Q_n = (F_n \mid M_{-\infty})$, $P_n \perp Q_n$ *for each* n *;*

(ii) $P_n = \sum_{k=0}^{\infty} P_{n,k}$ *where* $P_{n,k} \in \sigma(G_{n-k})$ *and further*

$$P_{n+f,k} = U^f P_{n,k}$$
$$\langle G_{n-k}, P_{n,k} \rangle = \langle G_{-k}, P_0 \rangle = \langle G_{-k}, F_0 \rangle$$

(iii) *If* $S = \langle G_0, G_0 \rangle$ *is invertible, then* $P_{n,k} = G_{n-k} A_k$ *where* $A_k \in B(K)$ *and is independent of* n. *Moreover,*

$$SA_k = \langle G_{-k}, P_0 \rangle = \langle G_{-k}, F_0 \rangle, \qquad A_0 = I \text{ ;}$$

(iv) $\{Q_n\}_{-\infty}^{+\infty}$ *is purely deterministic and* $\sigma(Q_n)_{-\infty}^{k} = M_{-\infty}$ *for each* k.

Proof.

(i) Since $G_m \perp G_n$ if $m \neq n$, it is obvious that $M_{-\infty} \perp N_n$ and $M_n = M_{-\infty} \oplus N_n$. Therefore by (9), we have

$$\begin{aligned} F_n &= (F_n \mid M_n) = (F_n \mid M_{-\infty} \oplus N_n) \\ &= (F_n \mid N_n) + (F_n \mid M_{-\infty}) \\ &= P_n + Q_n \end{aligned}$$

say evidently $P_n \perp Q_n$ proving (i).

(ii) Since $P_n \in N_n$ by its definition and since $N_n = \sigma(G_k)_{-\infty}^{n}$, we may apply the Theorem 1(e) to get

$$P_n = \sum_{k=0}^{\infty} P_{n,k}, \qquad P_{n,k} \in \sigma(G_{n-k}),$$

further by the same theorem,

$$\begin{aligned} \langle G_{n-k}, P_{n,k} \rangle &= \langle G_{n-k}, P_n \rangle \\ &= \langle G_{n-k}, P_n + Q_n \rangle \\ &= \langle G_{n-k}, F_n \rangle \\ &= \langle G_{-k}, F_0 \rangle \end{aligned}$$

since $G_{n-k} \perp Q_n$ by stationarity. $P_{n+f,k} = U^f P_{n,k}$ is a clear consequence of stationarity.

(iii) follows by an identical argument from Theorem 1(e) bearing in mind that S is invertible.

(iv) is obvious since $Q_n \in M_{-\infty}$ for all n by definition. See (i) above.

Following Kolmogorov [2] we call the process $\{F_n\}_{-\infty}^{+\infty}$ ***regular*** if $(F_0 \mid M_{-n}) \to 0$ when $n \to \infty$.

Using arguments similar to [6], we can now given the following theorem:

Theorem 3. *The following statements about* $\{F_n\}_{-\infty}^{+\infty}$ *are equivalent:*

(i) $\{F_n\}_{-\infty}^{+\infty}$ *is regular;*

(ii) *There exists an orthogonal process* $\{H_n\}_{-\infty}^{+\infty}$ *such that* $\langle H_m, H_n \rangle = \delta_{mn} T$ *and further,*

$$F_n = \sum_{k=0}^{\infty} F_{n,k}$$

with $F_{n,k} \in \sigma(H_{n-k})$ *and* $U^j F_{n,k} = F_{n+j,k}$.

(iii) $M_{-\infty} = \{0\}$.

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