PREDICTION THEORY IN COMPLETE CORRELATED ACTIONS

Ramona VASIU
ROMANIAN-GERMAN UNIVERSITY OF SIBIU

***Abstract**:*
In this paper we present a prediction theory of an infinite process considered as time-evolution in the state space of a correlated action. We introduce the notion of a correlated action and construct its measuring space as an Aronszajn reproducing kernel Hilbert space. The fundamental notion of prediction theory such as stationary process, deterministic, white-noise and moving average processes are define in this context.

***Key words**: correlated action, process, white-noise*

***JEL classification:** C00*

Complete correlated actions

Let us consider the triplet $\{\mathrm{E},\mathrm{H},\Gamma\}$, where E is a separable Hilbert space, H is a right $L(\mathrm{E})$-module, and $\Gamma:\mathrm{H}\times\mathrm{H}\to L(\mathrm{E})$ verifies the following properties:

i) $\Gamma[h,h]\geq 0, \Gamma[h,h]=0 \Rightarrow h=0$,

ii) $\Gamma[h_{1,}h_2]^* = \Gamma[h_2,h_1]$,

iii) $\Gamma[\sum_i A_i h_i, \sum_j B_j g_j] = \sum_{i,j} A_i^* \Gamma[h_i, g_j] B_j$.

In (iii) one considered finite sums, and $\mathrm{A}h$ is in the meaning of $L(\mathrm{E})$-module H. The map $L(\mathrm{E})\times\mathrm{H}\to\mathrm{H}$ defined by

(1) $$(\mathrm{A},h)\to \mathrm{A}h$$

is called the *action* of $L(\mathrm{E})$ onto the *state space* H. The separable Hilbert space E is called the *parameter space* and the map $\mathrm{H}\times\mathrm{H}\to L(\mathrm{E})$ given by

(2) $$(h,g)\to\Gamma[h,g]$$

is called the *correlation* of the action of $L(\mathrm{E})$ onto H.

Such a triplet $\{\mathrm{E},\mathrm{H},\Gamma\}$ is called a *correlated action* of $L(\mathrm{E})$ onto H.

A simple example of correlated action can be constructed as follows. Let E,H be two separable Hilbert spaces and $\mathrm{H}=L(\mathrm{E},\mathrm{K})$. H becomes a right $L(\mathrm{E})$- module if we consider for $\mathrm{A}\in L(\mathrm{E})$ and $V\in L(\mathrm{E},\mathrm{K})$

(3) $$\mathrm{A}V = V\mathrm{A}$$

where $V\mathrm{A}$ is the usual composition of operators. We take the action of $L(\mathrm{E})$ onto H, $\Gamma:\mathrm{H}\times\mathrm{H}\to L(\mathrm{E})$ defined by

(4) $$\Gamma[V_1,V_2]=V_1^*V_2 .$$

Clearly Γ satisfies the properties (i) – (iii) and we obtain that $\{\mathrm{E},\mathrm{H},\Gamma\}$is a *correlated action* which is also called the *operator model*. As we shall now in the following theorem, any abstract correlated action can be imbedded into one of this type.

Theorem 1. *Let* $\{\mathrm{E}, \mathrm{H}, \Gamma\}$ *be a correlated action. There exist a Hilbert space* K *and an algebraic imbedding* $h \to V_h$ *of the right* $L(\mathrm{E})$ *- module* H *into the right* $L(\mathrm{E})$ *- module* $L(\mathrm{E}, \mathrm{K})$ *with the properties*

(5) $$\Gamma[h_1, h_2] = V_{h_1}^* V_{h_2} \qquad (h_1, h_2 \in \mathrm{H}).$$

The elements of the form

(6) $$\gamma_{(a,h)} = V_h a,$$

where $a \in \mathrm{E}$ *and* $h \in \mathrm{H}$ *span a dense subspace in* K.

This imbedding is unique up to a unitary equivalence.

Proof. Let $\Lambda = \mathrm{E} \times \mathrm{H}$ and $\gamma_{(a,h)}$ be the complex valued function defined on Λ by

(7) $$\gamma_{(a,h)}(b, g) = (\Gamma[g, h]a, b)_{\mathrm{E}}.$$

On the linear span of these function we define the sesquilinear form as follows. Consider

(8) $$\left\langle \sum_i c_i \gamma_{(a_i,h_i)}, \sum_j d_j \gamma_{(b_j,g_j)} \right\rangle = \sum_{i,j} c_i d_j \left(\Gamma[g_j, h_i] a_i, b_j \right).$$

For any $a_1, ..., a_n \in \mathrm{E}$, choose $a \in \mathrm{E}$ and $\mathrm{A}_j \in L(\mathrm{E})$ such that $\mathrm{A}_j a = a_j$.

For $h_1, ..., h_n \in \mathrm{H}$ we have

$$\left\langle \sum_i c_i \gamma_{(a_i,h_i)}, \sum_j c_j \gamma_{(a_j,h_j)} \right\rangle = \sum_{i,j} c_i \overline{c_j} \left(\Gamma[h_j, h_i] a_i, a_j \right)_{\mathrm{E}} =$$

$$= \sum_{i,j} c_i \overline{c_j} \left(\Gamma[h_j, h_i] A_i a, A_j a \right)_{\mathrm{E}} = \sum_{i,j} c_i \overline{c_j} \left(A_j^* \Gamma[h_j, h_i] A_i a, a \right)_{\mathrm{E}} =$$

$$= \left(\Gamma[\sum_j c_j A_j h_j, \sum_i c_i A_i h_i] a, a \right)_{\mathrm{E}} \geq 0.$$

Such a way $\langle .,. \rangle$ is a sesquilinear semi-positive definite form. The Hilbert space K follows in the usual way by this form. In fact K is the Aronsazjn reproducing kernel Hilbert space [1], [4].

For any $h \in \mathrm{H}$ we define

(9) $$V_h a = \gamma_{(a,h)} \qquad a \in \mathrm{E}.$$

We obtain a linear bounded operator V from E into K. Indeed we have

$$\|V_h a\|^2 = \|\gamma_{(a,h)}\|^2 = (\Gamma[h, h]a, a) \leq \|\Gamma[h, h]\| \cdot \|a\|^2.$$

For any $h_1, h_2 \in \mathrm{H}$ and $a, b \in \mathrm{E}$ we have

$$(\Gamma[h_1, h_2]a, b) = \left\langle \gamma_{(a,h_2)}, \gamma_{(a,h_1)} \right\rangle = \left\langle V_{h_2} a, V_{h_1} a \right\rangle = (V_{h_1}^* V_{h_2} a, a).$$

Therefore the property (5) is verified. The property (6) and the fact that the elements of the form $\gamma_{(a,h)} = V_h a$ span a dense subspace in K, result from the construction of the Hilbert space K.

If we have another imbedding of H into $L(\mathrm{E}, \mathrm{K})$ with the properties (5) and (6), let us denote it by $h \to V_{h'}$, then putting

(10) $$X V_{h'} a = V_h a$$

we obtain a unitary operator $X : \mathrm{K}' \to \mathrm{K}$ such that $X V_{h'} = V_h$.

That finished the proof.

The uniquely attached Hilbert space K to a correlated action $\{\mathrm{E}, \mathrm{H}, \Gamma\}$ is called the *measuring space* of the correlated action.

We say that a correlated action $\{\mathrm{E},\mathrm{K},\Gamma\}$ is a *complete correlated action* if the map $h \to V_h$ of H into $L(\mathrm{E},\mathrm{K})$ is surjective.

Stationary processes in complete correlated actions

By a Γ- *stationary process* in the correlated action $\{\mathrm{E},\mathrm{H},\Gamma\}$ we mean a sequence $\{f_n\}_{-\infty}^{+\infty}$ of elements in H such that $\Gamma[f_n, f_m]$ depends only on the difference $m-n$ and not on m and n separately. Concerning a Γ- stationary process $\{f_n\}_{-\infty}^{+\infty}$ we consider the following subspace of the measuring space of the correlated action $\{\mathrm{E},\mathrm{H},\Gamma\}$:

(11) $$\mathrm{K}_n^f = \bigvee_{-\infty}^{n} V_{f_n}\mathrm{E},$$

(12) $$\mathrm{K}_\infty^f = \bigvee_{-\infty}^{+\infty} V_{f_n}\mathrm{E}.$$

If we consider the linear manifold

(13) $$\mathrm{K}_n^f = \left\{ h \in \mathrm{H}; h = \sum_{k<n} A_k f_k,\ A_k \in L(\mathrm{E}) \right\}$$

then (11) can be expressed as

(14) $$\mathrm{K}_n^f = \bigvee_{h\in \mathrm{K}_n^f} V_h\mathrm{E}.$$

Let $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$ be two Γ- stationary processes. We say that $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$ are *cross-correlated* if $\Gamma[f_n, g_m]$ depends only on the difference $m-n$.

Theorem 2. *For any Γ- stationary processes $\{f_n\}_{-\infty}^{+\infty}$ there exists a unitary operator U_f on K_∞^f such that*

(15) $$V_{f_n} = U_f^n V_{f_0}.$$

The stationary processes $\{g_n\}_{-\infty}^{+\infty}$ is stationary cross-correlated with $\{f_n\}_{-\infty}^{+\infty}$ if there exists a unitary operator U_{fg} on $\mathrm{K}_\infty^{fg} = \mathrm{K}_\infty^f \vee \mathrm{K}_\infty^g$ such that

$$U_f = U_{fg} \,|\, \mathrm{K}_\infty^f \quad \textit{and} \quad U_g = U_{fg} \,|\, \mathrm{K}_\infty^g.$$

Proof. It is enough to define U_f on the generators of K_∞^f by

$$U_f V_{f_n} a = V_{f_{n+1}} a.$$

Then U_f define a unitary operator on K_∞^f which verifies (15).

Let $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$ be two Γ- stationary cross-correlated processes and U_f and U_g as above. If we put

(16) $$U_{fg}(V_{f_n}a + V_{g_m}b) = V_{f_{n+1}}a + V_{g_{m+1}}b,$$

then we have

$$\left\| U_{fg}(V_{f_n}a + V_{g_m}b) \right\|^2 = \left\| V_{f_{n+1}}a + V_{g_{m+1}}b \right\|^2 = \left\| \gamma_{(a,f_{n+1})} + \gamma_{(b,g_{m+1})} \right\|^2 =$$
$$= (\Gamma[f_{n+1}, f_{n+1}]a,a) + (\Gamma[g_{m+1}, g_{m+1}]b,b) + 2\,\mathrm{Re}(\Gamma[f_{n+1}, g_{m+1}]a,b) =$$
$$= (\Gamma[f_n, f_n]a,a) + (\Gamma[g_m, g_m]b,b) + 2\,\mathrm{Re}(\Gamma[f_n, g_m]a,b) =$$
$$= \ldots = \left\| V_{f_n}a + V_{g_m}b \right\|^2$$

It results that (16) defines unitary operator U_{fg} which verifies the requested properties.

This finished the proof.

The unitary operator U_f is called the *shift operator* attached to the stationary processes $\{f_n\}_{-\infty}^{+\infty}$ and U_{fg} the *extended shift* of the Γ- stationary cross-correlated processes $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$.

In what follows, for a Γ- stationary processes $\{f_n\}_{-\infty}^{+\infty}$ we write V_f for V_{f_0}, and by (15) we have

$$\mathrm{K}_\infty^f = \bigvee_{-\infty}^{+\infty} U_f^n V_f \mathrm{E}.$$

It is easy to verify that the map $n \to \Gamma_f(n) = \Gamma[f_0, f_n]$ is a $L(\mathrm{E})$- valued positive definite function on the group Z. This function is called the *cross-correlation function*, or the *correlation function* of the Γ- stationary processes $\{f_n\}_{-\infty}^{+\infty}$. Also, for the cross-correlated Γ- stationary processes $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$ there exists the *cross-correlation function* defined by $n \to \Gamma_{fg}(n) = \Gamma[f_p, g_{p+n}]$.

Now we give some definition concerning the Γ- stationary process $\{g_n\}_{-\infty}^{+\infty}$ is called *white noise process,* provided $\Gamma[g_n, g_m] = 0$ for $n \neq m$.

We say that Γ- stationary process $\{f_n\}_{-\infty}^{+\infty}$ contains the white noise process $\{g_n\}_{-\infty}^{+\infty}$ if:

(17) (i) $\{g_n\}_{-\infty}^{+\infty}$ is stationary cross-correlated with $\{f_n\}_{-\infty}^{+\infty}$ and $\Gamma[f_n, g_m] = 0$ for $m > n$;

(ii) $V_g \mathrm{E} \subset \mathrm{K}_0^f$;

(iii) $\mathrm{Re}\,\Gamma[f_n - g_n, g_n] \geq 0$.

The Γ- stationary process $\{f_n\}_{-\infty}^{+\infty}$ is called *deterministic* if it contains no non-zero white noise process.

The Γ- stationary process $\{f_n\}_{-\infty}^{+\infty}$ is called a *moving average* of a white noise $\{g_n\}_{-\infty}^{+\infty}$ if $\{f_n\}_{-\infty}^{+\infty}$ contains $\{g_n\}_{-\infty}^{+\infty}$ and $\mathrm{K}_\infty^g = \mathrm{K}_\infty^f$.

Theorem 3. *(Wold decomposition) The Γ- stationary process $\{f_n\}_{-\infty}^{+\infty}$ admits a unique decomposition of the form*

(18) $$f_n = u_n + v_n,$$

where $\{u_n\}_{-\infty}^{+\infty}$ is a moving average of a white noise process $\{g_n\}_{-\infty}^{+\infty}$ contained in $\{f_n\}_{-\infty}^{+\infty}$, $\{v_n\}_{-\infty}^{+\infty}$ is a deterministic process and

(19) $$\Gamma[u_n, v_n] = 0 \quad n, m \in Z.$$

The white noise $\{g_n\}_{-\infty}^{+\infty}$ is the maximal white noise process contained in $\{f_n\}_{-\infty}^{+\infty}$.

Proof. If we write

(20) $$\mathrm{K}_+ = \bigvee_0^{+\infty} U_f^{*n} V_f \mathrm{E},$$

then $\mathrm{K}_+ = \mathrm{K}_0^f$ and $\mathrm{K}_n^f = U^n \mathrm{K}_0^f$. Putting

(21) $$U_+ = U_f^* \mid K_+$$

then U_+ is an isometric operator on K_+ and we can consider its Wold decomposition

(22) $$K_+ = M_+(F) \oplus \Re ,$$

where $F = K_+ \Theta U_+ K_+$, $M_+(F) = \bigoplus_0^\infty U_+^n F$ and $\Re = \bigcap_{n\geq 0} U_+^n K_+$.

Let P be the orthogonal projection of K_+ onto $M_+(F)$ and P_F be the orthogonal projection of K_+ onto the wandering subspace F. Using the imbedding $h \to V_h$ of H into $L(E, K)$ and we can consider

(23) $$f_n = U_f^n V_f .$$

If we put $u_n = U_f^n P V_f$, $v_n = U_f^n (I - P) V_f$ and $g_n = U_f^n P_F V_f$, then (18) is obvious and we have

$$\Gamma[u_n, v_n] = V_f^* P U_f^{m-n} (I-P) V_f = V_f^* U_f^{m-n} P (I-P) V_f = 0 .$$

Hence (19) is verified.

Because

$$\Gamma[g_n, g_m] = V_f^* P_F U_f^{m-n} P_F V_f = 0, \quad n \neq m,$$

it results that $\{g_n\}_{-\infty}^{+\infty}$ is a white noise process. The Γ- stationary white noise process $\{g_n\}_{-\infty}^{+\infty}$ is contained in $\{u_n\}_{-\infty}^{+\infty}$. Indeed, we have:

(i) $\{g_n\}_{-\infty}^{+\infty}$ is Γ- stationary cross-correlated with $\{u_n\}_{-\infty}^{+\infty}$ and $\Gamma[u_n, g_n] = V_f^* P U_f^{m-n} P_F V_f = 0$ for $m > n$;

(ii) $V_g E = P_F V_f E \subset P V_f E \subset K_+^u$;

(iii) $\Gamma[u_n - g_n, g_n] = \Gamma[u_n, g_n] - \Gamma[g_n, g_n] = V_f^* P P_F V_f - V_f^* P_F V_f = 0$.

Therefore (17) are verified and $\{g_n\}_{-\infty}^{+\infty}$ is a white noise contained in $\{u_n\}_{-\infty}^{+\infty}$.

Since we clearly have

(24) $$K_\infty^g = K_\infty^u = M(F)$$

it follows that the process $\{u_n\}_{-\infty}^{+\infty}$ is a moving average of the white noise $\{g_n\}_{-\infty}^{+\infty}$.

To see that the white noise $\{g_n\}_{-\infty}^{+\infty}$ is also contained in the Γ- stationary process $\{f_n\}_{-\infty}^{+\infty}$, we have for any $a \in E$ and $m > n$

$$(\Gamma[f_n, g_n]a, a) = (V_f^* U_f^{m-n} P_F V_f a, a) = (P_F V_f a, U_+^{m-n} V_f a) = 0 .$$

Also we have $V_g E = P_F V_f E \subset K_+^f$ and

$$\Gamma[f_n - g_n, g_n] = \Gamma[f_n, g_n] - \Gamma[g_n, g_n] = V_f^* P_F V_f - V_f^* P_F^2 V_f = 0 .$$

Hence (17) are again satisfied.

Let $\{g_n'\}_{-\infty}^{+\infty}$ be another white noise process contained in $\{f_n\}_{-\infty}^{+\infty}$. We shall see that $\{g_n'\}_{-\infty}^{+\infty}$ is contained in $\{g_n\}_{-\infty}^{+\infty}$ too. Firstly we see that

(25) $$V_{g'} E \subset F .$$

Indeed, for any $a, a_n \in E$ we have

$$(V_{g'} a, U_f^{*n+1} V_f a_n)_K = (V_f^* U_f^{n+1} V_{g'} a, a_n)_K = (\Gamma[f_0, g_{n+1}'] a, a_n)_E = 0$$

because $\{g_n'\}_{-\infty}^{+\infty}$ is contained in $\{f_n\}_{-\infty}^{+\infty}$.

Using (25) and the fact that $\{g'_n\}_{-\infty}^{+\infty}$ is contained in $\{f_n\}_{-\infty}^{+\infty}$ it follows that (17) are verified, therefore $\{g'_n\}_{-\infty}^{+\infty}$ is contained in $\{g_n\}_{-\infty}^{+\infty}$, i.e. $\{g_n\}_{-\infty}^{+\infty}$ is the maximal white noise contained in $\{f_n\}_{-\infty}^{+\infty}$.

To see that $\{v_n\}_{-\infty}^{+\infty}$ is a deterministic process is sufficient to consider a white noise process $\{l_n\}_{-\infty}^{+\infty}$ contained in $\{v_n\}_{-\infty}^{+\infty}$. Then it is easy to verify that $\{l_n\}_{-\infty}^{+\infty}$ is contained in $\{f_n\}_{-\infty}^{+\infty}$ and by the maximality of $\{g_n\}_{-\infty}^{+\infty}$ in $\{f_n\}_{-\infty}^{+\infty}$ it follows that $\{l_n\}_{-\infty}^{+\infty}$ is contained in $\{g_n\}_{-\infty}^{+\infty}$. We have than

$$\Gamma[g_n, l_n] = V_f^* P_F V_l = V_f^* P_F (I-P) V_l = 0$$

and

$$\Gamma[l_n, l_n] = \operatorname{Re}\Gamma[g_n, l_n] - \operatorname{Re}\Gamma[g_n - l_n, l_n] \le 0,$$

which implies that $l_n = 0$.

If we consider

(26) $$f_n = u'_n + v'_n$$

another decomposition of the form (18) and (19), where $\{u'_n\}_{-\infty}^{+\infty}$ is a moving average of the white noise $\{g'_n\}_{-\infty}^{+\infty}$ contained in $\{f_n\}_{-\infty}^{+\infty}$, then by the maximality of $\{g_n\}_{-\infty}^{+\infty}$ it follows that $\{g'_n\}_{-\infty}^{+\infty}$ is contained in $\{g_n\}_{-\infty}^{+\infty}$ and it is easy follows that

(27) $$V_{g'}\mathrm{E} \subset F.$$

We shall see that in fact we have $\overline{V_{g'}\mathrm{E}} = F$.

From (26) we have

(28) $$V_f = V_{u'} + V_{v'}$$

and

(29) $$\mathrm{K}_\infty^f = \mathrm{K}_\infty^{u'} \oplus \mathrm{K}_\infty^{v'}.$$

Let us denote by $F_1 = F \Theta \overline{V_{g'}\mathrm{E}}$ and $q_n = U_f^n P_{F_1} V_f$. It is easy to see that $\{q_n\}_{-\infty}^{+\infty}$ is a white noise process contained in $\{f_n\}_{-\infty}^{+\infty}$ and using the fact that $V_q\mathrm{E} \perp \mathrm{K}_+^u$ and (29) we can verify that the relation (17) concerning $\{v'_n\}_{-\infty}^{+\infty}$. Therefore $\{q_n\}_{-\infty}^{+\infty}$ is a white noise contained in the deterministic process $\{v'_n\}_{-\infty}^{+\infty}$, i.e. $q_n = 0$. Hence $F_1 = \{0\}$ and consequently $\overline{V_{g'}\mathrm{E}} = F$. We obtain therefore that $\mathrm{K}_\infty^{u'} = \mathrm{K}_\infty^{g'} = M(F)$, $\mathrm{K}_\infty^{v'} = \Re$ and by (28), (29) it follows that $V_{u'} = PV_f$. So we have $u' = u$ and $v' = v$.

The proof of the theorem is finished.

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